Journal of Applied Mechanics and Technical Physics, Vol. 42, No. 4, pp. 696–700, 2001

NUMERICAL ANALYSIS OF BRANCHED SHAPES OF ARCHES IN BENDING

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UDC 539.370

Nonlinear boundary-value problems of plane bending of elastic arches under a uniformly distributed load are solved by the shooting method. The problems are formulated for a system of six first-order ordinary differential equations with a finite-rotation field independent of displacements. Simply supported and clamped cases are considered. Branching solutions of the boundary-value problems are obtained. For a simply supported arch, a set of solutions describes symmetric and nonsymmetric shapes of bending, which correspond to positive, negative, and zero loads. For a clamped arch, the set of solutions consists of symmetric shapes that occur only for positive loads.

A three-hinge truss model proposed by Mises [1] is one of the first models that reproduce the main specific features of nonlinear deformation of arches: the existence of several forms of equilibrium that correspond to the same load and instability "in the large" (jumplike transition from one form of equilibrium to another). Timoshenko [2, 3] studied these specific features by solving a nonlinear problem of bending of a shallow simply supported arch loaded by uniform normal pressure (an approximate solution was obtained by the energy method). Later, Vol'mir [4] considered linearized problems of arch stability. With the advent of computers, the step-by-step (with respect to the loading parameter) method for solving boundary-value problems of deformation of structures has seen widespread use [5]. This method is used to find solutions branching from the basic solution; to this end, one has to resort to special techniques of constructing solutions in the neighborhood of bifurcation points. This makes it difficult to find isolated solutions typical of nonlinear problems of deformation of rods, plates, and shells. In the present study, the shapes of arch bending are analyzed by the shooting method, which reduces the nonlinear boundary-value problem to a finite set of nonlinear Cauchy problems.

System of Equations. Using the Cartesian coordinate system x_j with the orthonormal basis $\boldsymbol{e}_j$ ($j = 1$, 2, 3), the base line of a circular arch is defined by the parametric equations

$$x_1 \equiv 0, \qquad x_2 = r(\cos(\alpha t) - \cos\alpha), \qquad x_3 = r\sin(\alpha t) \qquad \forall\ t \in [-1, 1],$$

where t is the internal parameter of the line, r is the radius of the line, and 2α is the opening angle of the arch. The arch has a uniform cross section (profile) A, and the base line passes through the cross-sectional centroid. We consider plane bending of the arch under a load distributed along its length and specified by the vector

$$\boldsymbol{P} = P_2\boldsymbol{e}_2 + P_3\boldsymbol{e}_3. \tag{1}$$

We use the nonlinear equations of the one-dimensional model of a deformable rod [6] to determine the forms of equilibrium of the bent arch in the form

$$x_1 \equiv 0, \qquad x_2 = y(t), \qquad x_3 = z(t), \tag{2}$$

where y and z are desired functions. The arch material is assumed to be transversely isotropic and linearly elastic.

On the interval $(-1, 1)$, the nonlinear problem of plane bending of the arch is formulated as a system of six ordinary differential equations [7]

$$\begin{gathered} y_0' = y_1 + \alpha, \qquad y_1' = f_2 - (\gamma - 1)\varepsilon^2 f_2 f_3, \qquad y_2' = \varepsilon^2(\gamma f_2 \cos y_0 - f_3 \sin y_0) - \sin y_0, \\ y_3' = \varepsilon^2(\gamma f_2 \sin y_0 + f_3 \cos y_0) + \cos y_0, \qquad y_4' = -p_2, \qquad y_5' = -p_3, \\ f_2 \equiv y_4 \cos y_0 + y_5 \sin y_0, \qquad f_3 \equiv -y_4 \sin y_0 + y_5 \cos y_0 \end{gathered} \tag{3}$$

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Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 42, No. 4, pp. 155–160, July–August, 2001. Original article submitted October 30, 2000; revision submitted January 24, 2001.

0021-8944/01/4204-0696 $25.00

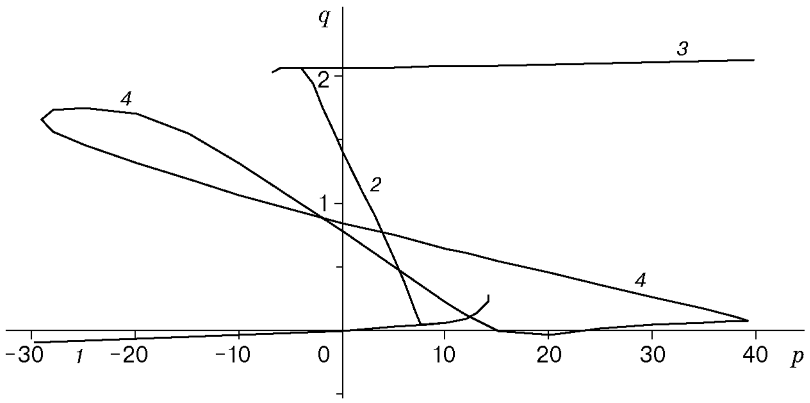

Fig. 1

with six unknown functions

$$y_0 = \theta, \quad y_1 = Yl/H, \quad y_2 = y/l, \quad y_3 = z/l, \quad y_4 = X_2 l^2/H, \quad y_5 = X_3 l^2/H \tag{4}$$

and parameters

$$\alpha = l/r, \qquad \gamma = E/G, \qquad \varepsilon^2 = I/(Al^2), \qquad p_j = P_j l^3/H, \qquad H = EI. \tag{5}$$

In (3)–(5), $\theta(t)$ is the angle of inclination of the arch cross section to the x_2 axis, $Y(t)$ is the bending moment about the x_1 axis, $y(t)$ and $z(t)$ are the coordinates of the point t in the (x_2, x_3) plane, $X_2(t)$ and $X_3(t)$ are the Cartesian components of the force vector, $2l$ is the length of the line, G is the shear modulus, E is the modulus of longitudinal extension and compression, and I is the cross-sectional moment of inertia about the x_1 axis; the prime denotes differentiation with respect to t. System (3) describes the nonlinear elastic bending of a circular arch for given parameters of loading p_2 and p_3, stiffness parameters α, γ, and ε, and fixing conditions of the arch.

Simply Supported Arch under Gravity Pressure. We understand the gravity pressure as a load uniformly distributed along the arch and directed parallel to the x_2 axis. In the course of deformation, the vector $\boldsymbol{P}(-P, 0)$ (1) does not rotate (P is the load intensity per unit length of the arch). The functions p_2 and p_3 in (3) take the values $p_2 = -p$ and $p_3 = 0$, where $p = Pl^3/H$ is the normalized parameter of pressure. We consider an arch simply supported at the boundary points $t = -1$ and $t = 1$, where the bending moments and displacements vanish:

$$y_1(\mp 1) = 0, \qquad y_2(\mp 1) = 0, \qquad y_3(\mp 1) = \mp(1/\alpha)\sin\alpha. \tag{6}$$

The nonlinear boundary-value problem (3), (6) is solved by the shooting method: at the point $t = -1$, we specify six conditions

$$\begin{gathered} y_1(-1) = 0, \qquad y_2(-1) = 0, \qquad y_3(-1) = -(1/\alpha)\sin\alpha, \\ y_0(-1) = k_1, \qquad y_4(-1) = k_2, \qquad y_5(-1) = k_3 \end{gathered} \tag{7}$$

and vary the parameters k_j to construct a three-parameter family of the solutions $\boldsymbol{y}(t, k_j)$ of the one-point problem (3), (7) ($\boldsymbol{y}$ is the vector of desired functions). The varied parameters corresponding to the solution of the initial boundary-value problem (3), (6) are determined iteratively with the use of three conditions (6) specified at the boundary point $t = 1$. This procedure is employed for a fixed parameter p. To find states of equilibrium that do not refer to the basic branch of the solution, we use a modified procedure, where one of the boundary parameters k_j is fixed and the pressure parameter p is varied. The use of both procedures enables us to control the accuracy of the solution of the problem. Numerical solutions of the problems were obtained with the use of the Mathcad-7 software.

Figure 1 shows the kinematic parameter q (displacement of the point $t = 0$ along the x_2 axis) versus the pressure parameter p for an arc with parameters $\alpha = \pi/4$, $\gamma = 2.5$, and $\varepsilon = 0.02$. It should be noted that positive values of p and q correspond to the load and displacement opposite in direction to the x_2 axis. Curve 1 is the branch of the basic nonlinear shapes (modes) of bending, which are symmetric about the x_2 axis and little different from the linear shapes for $p < 5$. If no restrictions are imposed on the elasticity and strength of the arch, the basic modes

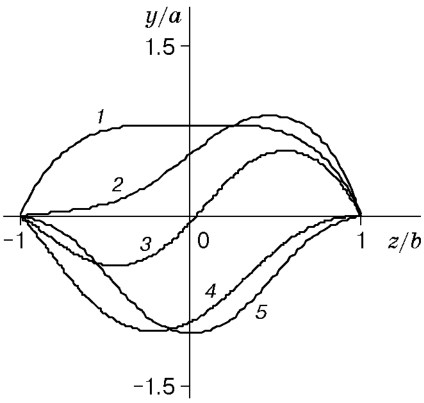

Fig. 2

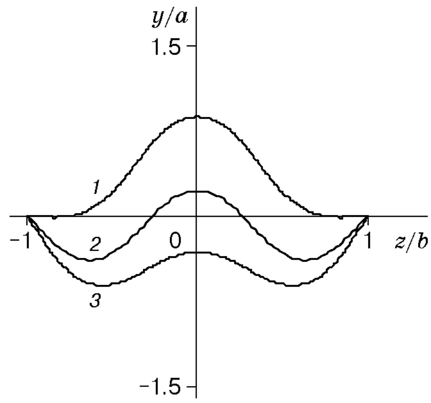

Fig. 3

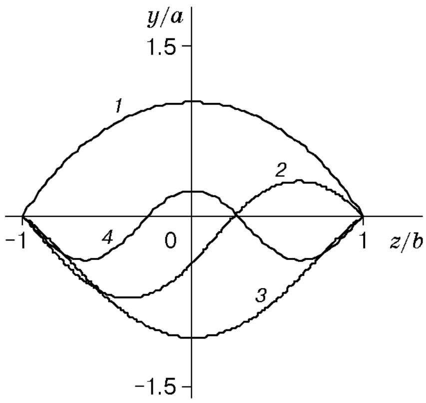

Fig. 4

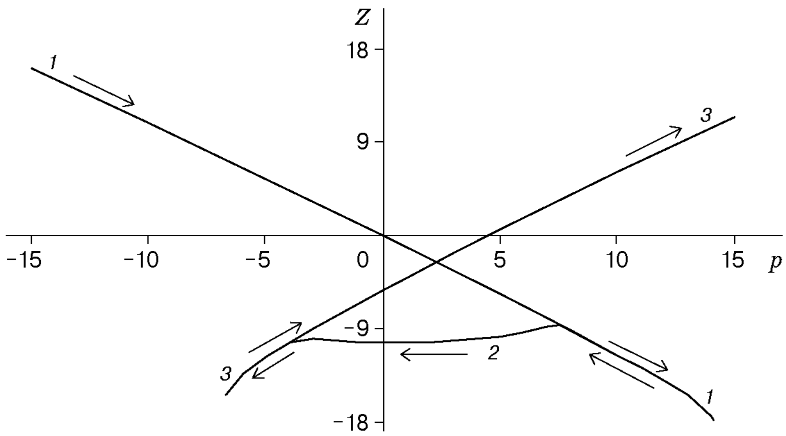

Fig. 5

occur in the semi-infinite interval $-\infty < p \leqslant p^+$, where $p^+ \approx 14$ is the upper critical point. The branch contains the point ($p = 0$, $q = 0$) that corresponds to the initial (stress-free) configuration of the arch. Curve 2 is the branch of nonsymmetric configurations; it connects the basic branch and the branch of snapped-through symmetric modes (curve 3). The latter branch exists in the semi-infinite interval $p^- \leqslant p < +\infty$, where $p^- \approx -6.7$ is the lower critical point.

Figure 2 shows the equilibrium configurations of the arch corresponding to branches 1–3 (see Fig. 1) in the coordinates z/b and y/a (a and $2b$ are the height and span of the arch, respectively). Curves 1–5 in Fig. 2 correspond to the points (p, q) with the coordinates (14, 0.234), (6, 0.381), (2.46, 1), (−3, 1.94), and (−6.7, 2.03), respectively. Curve 1 is the basic mode corresponding to the upper critical point. Curves 2–4 are nonsymmetric transitional modes corresponding to the second branch. Configuration 5 (snapped-through) corresponds to the lower critical point of the dependence $q(p)$.

In addition to bending modes shown in Fig. 2, there are symmetric and nonsymmetric highly oscillating modes. Curve 4 in Fig. 1 is the branch of symmetric modes with three or five half-waves. These modes are smoothly transformed one into the other, as is demonstrated by the closed curve 4. Figure 3 shows some modes with three half-waves. Curves 1–3 refer to the points (p, q) with the coordinates (12, 0.121), (0, 0.777), and (−10, 1.311), respectively.

The equilibrium configurations shown in Fig. 4 can occur for zero pressure ($p = 0$). Curves 1–4 refer to the points (p, q) with the coordinates (0, 0), (0, 1.41), (0, 2.066), and (0, 0.777), respectively. In contrast to the initial configuration 1, modes 2–4 are stressed and balanced by the force $X_3 = HZ/l^2$ that occurs at the supported points. These modes correspond to the points of intersection of the branches with the q axis (see Fig. 1). Figure 5 shows the parameter of the support force Z versus the pressure parameter (curve 1 refers to the basic modes of bending, curve 2 to nonsymmetric transitional modes, and curve 3 to snapped-through modes).

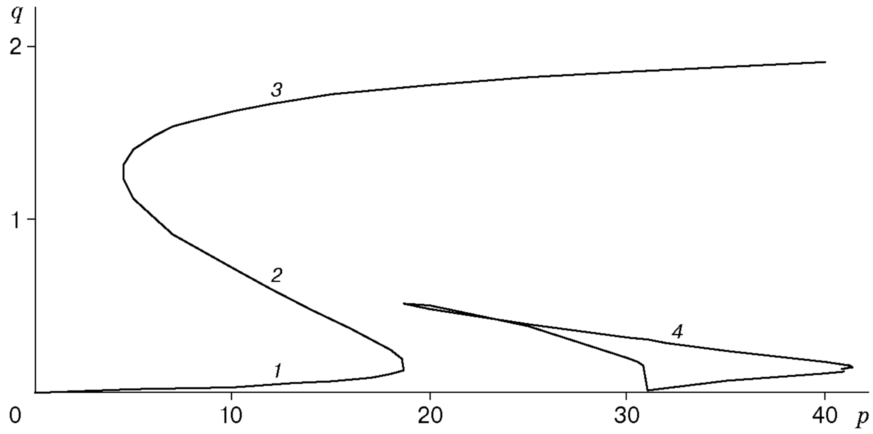

Fig. 6

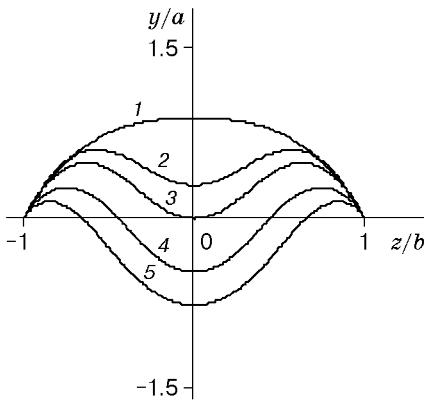

Fig. 7

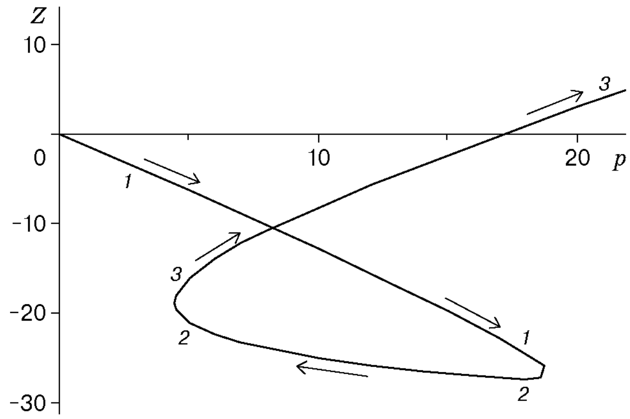

Fig. 8

The relation $q(p)$ obtained for the simply supported arch (see Fig. 1) shows that the basic shapes of equilibrium become unstable with respect to small perturbations when the load parameter reaches the value $p \approx 7.5$. The basic configuration of the arch snaps to the buckled configuration (point at branch 3). The backward snap (from branch 3 onto branch 1) becomes possible for $p < -4$.

Clamped Arch Under Normal Pressure. The normal pressure is a follower load. Let P be the pressure intensity per unit length of the arch. When the arch is bent, the vector $\boldsymbol{P}$ [see Eq. (1)] is directed along the normal to the base line [Eqs. (2)]. In this case, we have

$$P_2 = -P\cos\theta, \qquad P_3 = -P\sin\theta, \qquad p_2 = -p\cos y_0, \qquad p_3 = -p\sin y_0. \tag{8}$$

In the numerical solution, the parameter $p = Pl^3/H$ is assumed to be constant (uniform pressure); for $p > 0$, the vector $\boldsymbol{P}$ (1) is directed along the inward normal. In contrast to conditions (6), the boundary conditions for the arch with clamped ends (without rotations and displacements) are given by

$$y_0(\mp 1) = \mp\alpha, \qquad y_2(\mp 1) = 0, \qquad y_3(\mp 1) = \mp(1/\alpha)\sin\alpha. \tag{9}$$

Figures 6–8 show the results of the numerical solution of the nonlinear boundary-value problem (3), (8), (9) by the shooting method. Figure 6 shows the parameter of state q (vertical displacement of the central point) on the pressure parameter p: branch 1 refers to the basic modes of bending, branch 2 to the transitional modes, branch 3 to the snapped-through modes, and branch 4 to modes with four and five half-waves. The branches are separated by critical points [extrema of the relation $p(q)$]. Figure 7 shows the configurations of the arch for transition from one branch to another: curve 1 is the basic mode for the critical load ($p \approx 18.5$ and $q = 0.125$), curves 2 and 3 are transitional (unstable) modes (curve 2 refers to $p = 10$ and $q = 0.716$ and curve 3 to $p = 6$ and $q = 1.01$), curves 4 and 5 are snapped-through modes (curve 4 refers to $p = 6$ and $q = 1.48$ and curve 5 to $p = 20$ and $q = 1.77$). As in the case of a simply supported arch, modes with four and five half-waves correspond to the closed curve 4 in the

(p, q) plane (see Fig. 6). In contrast to the simply supported arch, this curve is located to the right of the upper critical point on the basic branch, and all configurations are symmetric. Nonsymmetric solutions were not found in this boundary-value problem. For zero pressure, only the stress-free state of the arch is possible; for $p < 0$, only the basic modes occur. Figure 8 shows the support reaction Z versus p; this curve consists of three portions, which correspond to those shown in Fig. 6.

Conclusions. The results obtained show that the shooting method can be used to solve nonlinear boundary-value problems of arch bending. The set of solutions depends qualitatively on the type of boundary conditions. In the simply supported case, the set of solutions is wider than that in the clamped case. In both cases, relations between the parameter of state q and the load parameter p are not monotonic and admit a catastrophe, i.e., instantaneous jump from the basic to the snapped-through configuration. An analysis of the problems in which free displacements of the boundary points are allowed along the x_3 axis (in this case, $y_5 \equiv 0$) gives different results: the corresponding dependences are monotonic, i.e., the catastrophe is impossible. At the same time, when the character of the load is changed (from translational to follower and vice versa), no qualitative change in the solutions is observed; moreover, the results are close quantitatively. In addition to the lower modes of bending, we found multiwave modes characterized by higher levels of elastic energy. These modes can occur under shock loading of arches.

The above-considered results are also valid for cylindrical panels with free curvilinear rims.

REFERENCES

1. R. Mises, “”Uber die Stabilitätsprobleme der Elastizitätstheorie,” *Z. Angew. Math. Mech.*, **3**, 406–462 (1923).
2. S. P. Timoshenko, “Buckling of flat curved bars and slightly curved plates,” *Trans. ASME, Ser. E, J. Appl. Mech.*, **2**, 17–20 (1935).
3. S. P. Timoshenko, *Stability of Rods, Plates, and Shells* [Russian translation], Nauka, Moscow (1971).
4. A. S. Vol’mir, *Stability of Deformable Systems* [in Russian], Nauka, Moscow (1967).
5. E. Riks, “An incremental approach to the solution of snapping and buckling problems,” *Int. J. Solids Struct.*, **15**, 529–551 (1979).
6. L. I. Shkutin, “Generalized models of the Cosserat type for finite deformation analysis of thin bodies,” *Prikl. Mekh. Tekh. Fiz.*, **37**, No. 3, 120–132 (1996).
7. L. I. Shkutin, “Numerical analysis of branching modes of bending of rods,” *Prikl. Mekh. Tekh. Fiz.*, **42**, No. 2, 141–147 (2001).